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THE PROBLEM OF STABILIZING THE STEADY MOTIONS OF SYSTEMS WITH CYCLIC COORDINATES*

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A new approach, based on linear control theory, is used to the study the stabilization of steady motions of systems in which only the cyclic coordinates are controllable /1, 2/. Unlike techniques previously used to solve this problem /3, 4/, which require the reduced system to have an asymptotically stable invariant manifold, maximum use is made here of the control possibilities inherent in the system. Several new controllability and observability criteria are formulated, taking the structure of the forces acting on the system into account.

Some questions concerning the stabilization of steady solutions were considered in /5, 6/ for a given structure of the controls, and the stability of the closed-loop system was studied. Similar studies in /7, 8/ also analysed controllability conditions.

1. We consider a holonomic mechanical system with time-independent constraints, assuming that the generalized coordinates of the system, $q_1, \ldots, q_n$, include a group q_j $(j = r+1, \ldots, n,\ r < n)$ not occurring explicitly in the expression for the kinetic energy T of the system. We shall assume that the forces applied to the system are also independent of these coordinates, which are usually called pseudocyclic. The remaining coordinates q_i $(i = 1, \ldots, r)$ are called positional coordinates. Let $q, q^{\cdot}, \omega$ denote the column matrices whose elements are the positional coordinates, and the positional and pseudocyclic velocities, respectively.

In the general case the system is gyroscopically constrained and its kinetic energy is

$$T = {}^1/_2 q^{\cdot T} A(q)\, q^{\cdot} + q^{\cdot T} C(q)\, \omega + {}^1/_2 \omega^T B(q)\, \omega$$

Here A and B are positive definite symmetric matrices and C is a rectangular matrix. Their coefficients depend only on the positional coordinates.

The generalized forces corresponding to the positional coordinates are assumed to be known; each is the sum of potential and dissipative forces:

$$Q_i = \partial U/\partial q_i + Q_{id} \quad (i = 1, \ldots, r)$$

The generalized forces $F_j = F_j(q, q^{\cdot}, \omega)$ $(j = r+1, \ldots, n)$ corresponding to the pseudocyclic coordinates will be treated as controls, to be determined later.

Information about the values of $q, q^{\cdot}, \omega$ is obtained by a measurement $\Sigma = \Sigma(q, q^{\cdot}, \omega)$ of dimension $l \times 1$.

Suppose that under certain initial conditions the system admits of a steady motion:

$$q(t) = q_0 = \text{const}, \quad \omega(t) = \omega_0 = \text{const}$$

The quantities q_0, ω_0 are determined from the equations

$$-\partial U/\partial q_i - {}^1/_2 \partial \omega^T B \omega/\partial q_i = 0$$

Under these conditions $F_j = 0$.

If the equalities

$$\partial U/\partial q_i = \partial B_{kj}/\partial q_i = 0 \quad (i = 1, \ldots, r) \tag{1.1}$$

hold at $q = q_0$, where B_{kj} are the coefficients of the matrix B, the corresponding steady motion is said to be trivial; any other steady motion will be called essential.

Introducing notation for the differences $x = q - q_0$, $\eta = \omega - \omega_0$, we write the Lagrange equations in terms of x, η in matrix notation, separating out the linear terms:

$$\begin{gathered} A_0 x^{\cdot\cdot} + (D_0 + G_0)\, x^{\cdot} + W_0 x + C_0 \eta^{\cdot} - P_0^T \eta = X(x, x^{\cdot}, \eta) \\ B_0 \eta^{\cdot} + C_0^T x^{\cdot\cdot} + P_0 x^{\cdot} = u + \Theta(x, x^{\cdot}, \eta) \end{gathered} \tag{1.2}$$

Here D_0 is the coefficient matrix of the linear part of the dissipative force vector Q_d, u

*Prikl. Matem. Mekhan., 53, 5, 707-714, 1989 is the linear part of the control vector F. A subscript zero means that the quantity in question is evaluated at $q = q_0$, $\omega = \omega_0$. The elements of the matrices G_0, W_0, P_0 are defined as follows:

$$G_{ij0} = \sum_{l=r+1}^{n} \left(\frac{\partial C_{il}}{\partial q_j} - \frac{\partial C_{jl}}{\partial q_i} \right) \omega_l \Big|_0, \quad P_{ij0} = \sum_{l=r+1}^{n} \left(\frac{\partial B_{lj}}{\partial q_i} \right) \omega_l \Big|_0$$

$$W_{ij0} = -\frac{\partial^2 U}{\partial q_i \partial q_j} - \frac{1}{2} \sum_{k,l=1}^{r} \omega_k \omega_l \frac{\partial^2 B_{kl}}{\partial q_i \partial q_j} \Big|_0$$

Finally, $X(x, x^{\cdot}, \eta)$, $\Theta(x, x^{\cdot}, \eta)$ are vector-valued functions containing terms non-linear in $x, x^{\cdot}, \eta$.

We consider the problem of stabilizing the steady motions q_0, ω_0 of the system by controls applied only with respect to the pseudocyclic coordinates, relying on the equations in the first approximation:

$$A_0 x^{\cdot\cdot} + (D_0 + G_0) x^{\cdot} + W_0 x + C_0 \eta^{\cdot} - P_0^T \eta = 0 \tag{1.3}$$

$$B_0 \eta^{\cdot} + C_0^T x^{\cdot\cdot} + P_0 x^{\cdot} = u$$

$$\sigma = H_0 x + L_0 x^{\cdot} + R_0 \eta \tag{1.4}$$

Here σ is the linear part of the measurement vector Σ, and H_0, L_0, R_0 are constant matrices of the appropriate dimensions.

To solve the stabilization problem one must first determine whether system (1.3) can in principle be stabilized, i.e., test the system for controllability. Second, one has to secure the necessary information about the state of the system (i.e., about the values of $x, x^{\cdot}, \eta$) this can be done by analysing the observability of system (1.3), (1.4). The third stage is to construct a stabilization algorithm, e.g., by introducing a feedback loop in the procedure for evaluating the state vector of the system, based on information derived from the measurements.

We will first state Kalman's necessary and sufficient conditions for observability and controllability /9/. To that end we reduce system (1.3), (1.4) to the Cauchy form:

$$y^{\cdot} = A_y y + B_y u, \quad \sigma = C_y y \tag{1.5}$$

$$y = (x, x^{\cdot}, \eta)^T, \quad C_y = (H_0, L_0, R_0)$$

$$A_y = \begin{Vmatrix} 0 & E_r & 0 \\ A_{y1} & A_{y2} & A_{y3} \\ A_{y4} & A_{y5} & A_{y6} \end{Vmatrix}, \quad B_y = \begin{Vmatrix} 0 \\ B_{y1} \\ B_{y2} \end{Vmatrix}$$

$$A_{y1} = -S_0^{-1} W_0, \quad A_{y2} = -S_0^{-1} (D_0 + G_0 - C_0 B_0^{-1} P_0), \quad A_{y3} = S_0^{-1} P_0^T$$

$$A_{y4} = M_0^{-1} C_0^T A_0^{-1} W_0, \quad A_{y5} = -M_0^{-1} (P_0 - C_0^T A_0^{-1} (D_0 + G_0)),$$

$$A_{y6} = -M_0^{-1} C_0^T A_0^{-1} P_0^T$$

$$B_{y1} = -S_0^{-1} C_0 B_0^{-1}, \quad B_{y2} = M_0^{-1}$$

$$S_0 = A_0 - C_0 B_0^{-1} C_0^T, \quad M_0 = B_0 - C_0^T A_0^{-1} C_0, \quad S_0 = S_0^T > 0,$$

$$M_0 = M_0^T > 0$$

E_r is the $r \times r$ identity matrix.

Theorem 1.1. System (1.5) is controllable and observable if and only if

$$\operatorname{rank} \| B_y, A_y B_y, \ldots, A_y^{n+r-1} B_y \| = n + r \tag{1.6}$$

$$\operatorname{rank} \begin{Vmatrix} C_y \\ C_y A_y \\ \cdots \\ C_y A_y^{n+r-1} \end{Vmatrix} = n + r$$

Verification of these conditions may be quite laborious.

2. To obtain effective conditions for the controllability of system (1.3), it will be convenient to study two cases separately: gyroscopically uncoupled systems (GUS; $C_0 \equiv 0$) and gyroscopically coupled systems (GCS).

A GUS in trivial steady motion (1.1) ($P_0 \equiv 0$) splits into two independent subsystems:

$$A_0 x^{\cdot\cdot} + D_0 x^{\cdot} + W_0 x = 0, \quad B_0 \eta^{\cdot} = u \tag{2.1}$$

of which only the second is controllable ($\det B_0 \neq 0$). Hence trivial steady motions of a GUS are not stabilizable. Nevertheless, subject to certain conditions on the matrices A_0, D_0, W_0, the trivial solution of system (2.1) may be asymptotically stable. A stability analysis of system (2.1) with an eye to the structure of the forces may be found in /3/.

We now consider the case of essential steady motions of a GUS.

The following theorems can be proved:

Theorem 2.1. A GUS of order $n+r$ is controllable if and only if the system

$$A_0x^{\cdot\cdot} + D_0x^{\cdot} + K_0x = P_0^TB_0^{-1}v, \quad K_0 = W_0 + P_0^TB_0^{-1}P_0 \tag{2.2}$$

of order $2r$ is controllable. An analogous theorem was stated in /8/.

Theorem 2.2. If the rank of P_0 is equal to the number of positional coordinates of the GUS $(\operatorname{rank} P_0 = r)$, the system is always controllable.

In this case it is obvious that the least number of controls is equal to the number of positional coordinats.

Corollary 2.1. A system with only one positional coordinate $(r=1)$ is controllable if and only if $P_0 \neq 0$.

There are various criteria that can be used to determine whether system (2.2) is controllable. Here we shall use a controllability criterion proposed in /10/, that takes the specific structure of the forces into consideration.

Theorem 2.3. A GUS is controllable if and only if

$$\operatorname{rank} \| \lambda^2A_0 + \lambda D_0 + K_0; \;\; P_0^TB_0^{-1} \| = r, \quad \forall\lambda \in \Lambda$$
$$\Lambda = \{\lambda_i\colon \det[\lambda^2A_0 + \lambda D_0 + K_0] = 0\}$$

Corollary 2.2. If $K_0 \equiv 0$, then the GUS is controllable if and only if $\operatorname{rank} P_0 = r$.

Example 2.1. Consider a physical pendulum of mass m, whose horizontal axis OO' may turn about a vertical axis NN' /3, 11/. The system has two degrees of freedom: rotation of the axis of oscillation through an angle ψ (pseudocyclic coordinate) and rotation of the body about the axis of oscillation through an angle ϑ (positional coordinate). The axes OO', NN' interest at O; the axes OO', OG_* (G_* is the centre of gravity of the body) are the principal axes of the inertia ellipsoid for the point O. Under these assumptions

$$2T = I_1\vartheta^{\cdot 2} + (I_2\sin^2\vartheta + I_3\cos^2\vartheta)\omega^2$$
$$U = mga\cos\vartheta, \quad a = |OG_*|, \quad \omega = \varphi^{\cdot}$$

Here I_1, I_2, I_3 are the moments of inertia of the body.

By Corollary 2.1, in the case of essential steady motions, this GUS is always controllable in the interval $0 \leqslant \vartheta \leqslant \pi$. The sole exception is the degererate case $\vartheta_0 = \pi/2$, which is possible only when $a=0$.

Example 2.2. Consider a heavy gyro in a perfect Cardan suspension, with the axis of rotation of the external gimbal vertical /3, 12/. The angle of nutation $\vartheta\,(0 \leqslant \vartheta \leqslant \pi)$ is a positional coordinate, the angular velocities of spin Ω and precession ω are pseudocyclic. The controls affecting the cyclic coordinates are the moment F_1 produced by a motor rotating the external gimbal, and the moment F_2 of a motor installed on the internal gimbal and driving the gyro itself. The essential steady motions are determined by the equation

$$(I_1 + J_2 - I_3 - J_3)\Omega_0^2\cos\vartheta_0 - I_3\omega_0\Omega_0 + mgz_0 = 0$$

Here I_1, I_3, J_2, J_3 are the moments of inertia of the rotor and the internal gimbal, m is the mass of the gyro, and z_0 $(z_0 \neq 0)$ is the distance from the centre of gavity of the gyro to the centre of the suspension.

In that case it can be shown, using Corollary 2.1, that this GUS is always controllable.

We now consider a GCS $(C_0 \not\equiv 0)$. The equations of the first approximation for the case of trivial steady motions of (1.1) $(P_0 \equiv 0)$ are

$$A_0x^{\cdot\cdot} + (D_0 + G_0)x^{\cdot} + W_0x + C_0\eta^{\cdot} = 0, \quad B_0\eta^{\cdot} + C_0^Tx^{\cdot\cdot} = u \tag{2.3}$$

It is obvious that if $C_0 \not\equiv 0$ system (2.3) does not split into two independent subsystems, as does (2.1) in the case of a GUS; instead, there is a cross connection through $\eta^{\cdot}$ and $x^{\cdot\cdot}$, which presents some additional possibilities of stabilization. This is particularly important when the matrix W_0 is not positive definite.

The following theorems can be proved.

Theorem 2.4. System (2.3) is controllable if and only if $\det W_0 \neq 0$ and the system

$$S_0x^{\cdot\cdot} + (D_0 + G_0)x^{\cdot} + W_0x = -C_0B_0^{-1}v \tag{2.4}$$

is controllable.

Theorem 2.5. System (2.3) is controllable if and only if

$$\det W_0 \neq 0, \quad \operatorname{rank} \| \lambda^2 S_0 + \lambda (D_0 + G_0) + W_0, \; C_0 B_0^{-1} \| = r,$$
$$\forall \lambda \in \Lambda_1, \quad \Lambda_1 = \{\lambda_i \quad \det [\lambda^2 S_0 + \lambda (D_0 + G_0) + W_0] = 0\}$$

Corollary 2.3. A system (2.3) with only one positional coordinate $r = 1$ is controllable if and only if $C_0 \neq 0$, $W_0 \neq 0$.

We now consider essential $(P_0 \not\equiv 0)$ steady motions of the GCS (1.3). The following theorems can be proved.

Theorem 2.6. System (1.3) is controllable if and only if the system

$$S_0 x^{\cdot\cdot} + N_0^T x^{\cdot} + K_0^T x = 0 \tag{2.5}$$
$$(N_0 = D_0 + G_0 - C_0 B_0^{-1} P_0 + P_0^T B_0^{-1} C_0^T)$$

is observable by the measurement

$$\sigma = C_0^T B_0^{-1} x^{\cdot} - P_0 B_0^{-1} x$$

Theorem 2.7. System (1.3) is controllable if and only if

$$\operatorname{rank} \left\| \begin{matrix} \lambda C_0^T - P_0 \\ \lambda^2 S_0 + \lambda N_0^T + K_0^T \end{matrix} \right\| = r, \quad \forall \lambda \in \Lambda_2$$
$$\Lambda_2 = \{\lambda_i \colon \det [\lambda^2 S_0 + \lambda N_0^T + K_0^T] = 0\}$$

Corollary 2.4. If $K_0 = W_0 + P_0^T B_0^{-1} P_0 = 0$, then system (1.3) is controllable if and only if $\operatorname{rank} P_0 = r$.

Corollary 2.5. If $r = 1$, i.e., the system has only one positional coordinate, then system (1.3) is controllable if and only if

$$\lambda C_0^T \neq P_0, \quad \forall \lambda \in \Lambda_2$$

Example 2.3. Consider a Cardan-suspended heavy gyro with directional asymmetry and vertical axis l_1 of rotation /4, 13/. The internal gimbal and rotor are linked by a cylindrical hinge l_0 with axis l_2 intersecting l_1 at a point O. Assume that the rotor axis l_3 is fixed in the body and the mass distribution of the rotor is symmetrical about this axis. Denote the centre of mass of the rotor, with coordinates x_1, y_1, z_1, by O_1

Choose moving coordinate frames $O\xi_1\eta_1\zeta_1$, $O\xi_2\eta_2\zeta_2$ and $O_1\xi_3\eta_3\zeta_3$. The axes $O\xi_1$, $O\xi_2$, $O_1\xi_3$ are respectively l_1, l_2, l_3. The plane $O\xi_1\eta_1$ contains the axis $O\xi_2$. The angle between $O\xi_1$ and $O\xi_2$ is denoted by $\varepsilon\ (0 < \varepsilon < \pi)$; λ, μ, ν are the direction cosines of the rotor in the system $O\xi_2\eta_2\zeta_2$,

$$I = \left\| \begin{matrix} A_2 & -G_2 & R_2 \\ -G_2 & B_2 & -D_2 \\ -R_2 & -D_2 & C_2 \end{matrix} \right\|$$

is the inertia tensor of the internal gimbal in the frame $O\xi_2\eta_2\zeta_2$, J is the moment of inertia of the external gimbal relative to l_1, A_* is the axial and B_* the equatorial moments of inertia of the rotor for the point O; m_2, m_3 are the masses of the internal gimbal and the rotor; x_2, y_2, z_2 are the coordinates of the common centre of mass of the internal gimbal and the rotor in the frame $O\xi_2\eta_2\zeta_2$ For our gyro $\nu = 0$, $z_2 = 0$ and

$$D_2 + m_3 y_1 z_1 = 0, \quad R_2 + m_3 z_1 x_1 = 0$$

The angle of nutation ϑ is a positional coordinate. We shall assume that $\vartheta = 0$ when l_3 lies in a plane parallel to l_1 and l_2 $(0 \leqslant \vartheta \leqslant \pi)$. The angles of precession ψ and spin φ are cyclic coordinates.

The generalized force corresponding to the positional coordinate ϑ is the sum of moments of the gravitational and dissipative forces $d\vartheta^{\cdot}$. The controls are the moment F_1 of the motor driving the external gimbal and the moment F_2 of the motor installed on the internal gimbal and driving the rotor.

The regular precession $\vartheta = 0$, $\omega_1 = \psi^{\cdot} = \text{const}$, $\omega_2 = \varphi = \text{const}$ is a trivial steady motion. Using Corollary 2.3, one can show that the GCS in question is controllable.

3. We now consider the problem of observability for system (1.3) using measurements

$$\sigma_1 = H_0 x + L_0 x^{\cdot} \tag{3.1}$$
$$\sigma_2 = R_0 \eta \tag{3.2}$$

where H_0, L_0 are constant $l \times r$ matrices, and R_0 a constant $l \times n - r$ matrix. The standard observability criterion of Theorem 1.1 is extremely laborious to apply if n is large. Thanks to the specific structure of the system, one can devise more effective observability conditions. Using the observability criterion of /14/, one can prove the following theorems.

Theorem 3.1. System (1.3), (3.1) is observable if and only if

$$\operatorname{rank}\left\|\begin{matrix} H_0 & 0 \\ -K_0 & P_0{}^T B_0^{-1}\end{matrix}\right\| = n \tag{3.3}$$

$$\operatorname{rank}\left\|\begin{matrix} H_0+\lambda L_0 \\ \lambda^2 S_0+\lambda N_0+K_0\end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda_3,\ \lambda\neq 0$$

$$\Lambda_3=\{\lambda_i\colon \det\,[\lambda^2 S_0+\lambda N_0+K_0]=0\}$$

Corollary 3.1. If $H_0\equiv 0$, system (1.3), (3.1) is not observable.

Corollary 3.2. If $L_0\equiv 0$, $H_0=E_r$, then system (1.3), (3.1) is observable if and only if $\operatorname{rank} P_0\geqslant n-r$. In particular, this means that for the system to be observable the number of positional coordinates must not be less than the number of cyclic coordinates.

Corollary 3.3. In the case of trivial steady motions $(P_0\equiv 0)$, system (1.3), (3.1) is not observable.

Corollary 3.4. In the case of a gyroscopically uncoupled system $(C_0\equiv 0)$, system (2.1), (3.1) is observable if and only if condition (3.3) holds and

$$\operatorname{rank}\left\|\begin{matrix} H_0+\lambda L_0 \\ \lambda^2 A_0+\lambda D_0+K_0\end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda,\ \lambda\neq 0$$

Theorem 3.2. System (1.3), (3.2) is observable if and only if

$$\det W_0\neq 0, \quad \operatorname{rank}\left\|\begin{matrix} P_0+\lambda C_0{}^T \\ \lambda^2 S_0+\lambda N_0+K_0\end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda_3,\quad \lambda\neq 0$$

Corollary 3.5. In the case of trivial steady motions $(P_0\equiv 0)$, system (1.3), (3.2) is observable if and only if

$$\det W_0\neq 0, \quad \operatorname{rank}\left\|\begin{matrix} \lambda C_0{}^T \\ \lambda^2 S_0+\lambda(D_0+G_0)+W_0\end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda_1,\ \lambda\neq 0$$

Corollary 3.6. The GUS $(C_0\equiv 0)$ (1.3), (3.2) is observable if and only if

$$\det W_0\neq 0, \quad \operatorname{rank}\left\|\begin{matrix} P_0 \\ \lambda^2 A_0+\lambda D_0+K_0\end{matrix}\right\| = r, \quad \forall\lambda\in\Lambda,\ \lambda\neq 0$$

Corollary 3.7. The GUS $(C_0\equiv 0)$ (1.3), (3.2) for trivial steady motions $(P_0\equiv 0)$ is not observable.

Corollary 3.8. If the GUS $(C_0\equiv 0)$ (1.3), (3.2) has $r=1$ (one positional coordinate), it is observable if and only if $W_0\neq 0$ and $P_0\neq 0$.

Corollary 3.9. In the case of trivial $(P_0\equiv 0)$ steady motions in system (1.3), (3.2) with $r=1$ (one positional coordinate), the system is observable if and only if $W_0\neq 0$, $C_0\neq 0$.

These results can be illustrated with reference to the examples considered in Sect.2.

Example 3.1. A physical pendulum is a GUS with one positional coordinate. Corollaries 3.1, 3.2 and 3.8 imply: 1) the system is not observable by the measurement $\sigma_4=x$, 2) the system is observable by the measurement $\sigma_5=x$ and observable by the measurement $\sigma_2=\eta$, except for the degenerate case in which the centre of mass coincides with the point of suspension $(a=0)$.

Example 3.2. For the Cardan-suspended gyro performing essential steady motions of the regular precession type, the system is not observable by the measurement $\sigma_4=x^{\boldsymbol{\cdot}}$ but observable by the measurements $\sigma_5=x$, $\sigma_2=\eta$, as is shown similarly by application of Corollaries 3.1, 3.2 and 3.8.

Example 3.3. Since regular precession of a Cardan-suspended heavy gyro with parallel asymmetry and vertical axis of rotation of the external gimbal corresponds to the case of trivial steady motions of a GCS, it follows from Corollaries 3.4 and 3.9 that the system is observable by the measurement $\sigma_2 = \eta$ but not observable by the measurements $\sigma_4 = x$, $\sigma_5 = x$

4. The third step in solving the stabilization problem is the construction of a stabilization algorithm. If the controllability conditions of Sect.2 are satisfied, this means that one can always select a control u in problem (1.3) by state feedback:

$$u = -K_1 x - K_2 x^{\cdot} - K_3 \eta \tag{4.1}$$

(here K_1, K_2, K_3 are constant matrices of suitable dimensions) in such a way as to obtain any preassigned roots of the characteristic equation of the closed-loop system:

$$\begin{gathered} A_0 x^{\cdot\cdot} + (D_0 + G_0) x^{\cdot} + W_0 x + C_0 \eta^{\cdot} - P_0 \eta = 0 \\ B_0 \eta^{\cdot} + C_0^T x^{\cdot\cdot} + P_0 x^{\cdot} = -K_1 x - K_2 x^{\cdot} - K_3 \eta \end{gathered} \tag{4.2}$$

Under these conditions the trivial solution of the full non-linear closed-loop Eq.(4.1) of system (1.2) will also be asymptotically stable.

The elements of the matrices K_1, K_2, K_3 may be determined in various ways. In particular, one can reduce the system to what is known as canonical controllable form, i.e., the initial system is split into several subsystems with scalar conrols /15/. Selection of control coefficients for these subsystems corresponding to preassigned damping factors presents no essential difficulties; for example, one can use the convenient procedure of /16/.

To produce a control (4.1) one needs all phase coordinates $x, x^{\cdot}, \eta$, which are usually measurable. However, as a study of the observability of the system described in Sect.3 will show, there is no need to measure all the phase coordinates. If the observability conditions stated in Sect.3 are satisfied, one can construct a stabilization algorithm for system (1.3) as

$$u = -K_1 x^{\circ} - K_2 x^{\circ\cdot} - K_3 \eta^{\circ} \tag{4.3}$$

where $x^{\circ}, x^{\circ\cdot}, \eta^{\circ}$ are estimates for the vectors $x, x^{\cdot}, \eta$, obtained from an estimation algorithm

$$y^{\circ\cdot} = A_y y^{\circ} + L_y (\sigma - C_y y^{\circ}) \tag{4.4}$$

where $y^{\circ T} = (x^{\circ}, x^{\circ\cdot}, \eta^{\circ})$, the matrices A_y, C_y are determined by (1.5), (1.6), and $\sigma = C_y y$ is a measurement by which system (1.5) is observable; the amplification coefficient matrix L_y is determined on the basis of any criterion for the estimation errors $\Delta y = y - y^{\circ}$ to be small. The closed-loop control system obtained in this case is described by Eqs.(1.3), (4.3) and (4.4)

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A UNIVERSAL SEQUENCE OF PERIOD-DOUBLING BIFURCATIONS OF THE FORCED OSCILLATIONS OF A PENDULUM*

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One of the most typical modes of chaotization in deterministic systems occurs when variation of the parameter characterizing the intensity of a disturbance takes a dynamical system through a sequence of period-doubling bifurcations from a regular to a stochastic mode of behaviour. The transition occurs in regions of phase space characterized by a strong local instability and obeys the law of universality recently discovered by Feigenbaum /1/.

In this paper continuation with respect to a parameter and methods of branching theory /3/ are used in combination to construct a sequence of period-doubling bifurcations for the forced oscillations of a conservative pendulum. This sequence is shown to possess the universality property.

1. We shall be concerned in this paper with the evolution and bifurcation of periodic solutions of the equation of forced oscillations of a pendulum

$$x^{\cdot\cdot} + k^2 \sin x = \lambda \sin \omega t \tag{1.1}$$

when the parameter λ is varied and with analysis of the stability of these solutions. Suppose that at some parameter value $\lambda = \lambda_{(0)}$ the pendulum oscillates with period $T = 2\pi/\omega$. The corresponding T-periodic solution ("T-solution") of Eq.(1.1) satisfies the conditions

$$x_{(0)}(0) = x_{(0)}(T), \quad x_{(0)}^{\cdot}(0) = x_{(0)}^{\cdot}(T) \tag{1.2}$$

The solutions of Eq.(1.1) are continuous functions of the initial conditions and the parameter λ, so that the T-periodicity conditions can be written

$$\begin{gathered} x_0 = x\ (x_0, x_0^{\cdot}, \lambda, T) \\ x_0^{\cdot} = x^{\cdot}(x_0, x_0^{\cdot}, \lambda, T) \quad (x_0 = x(0)) \end{gathered} \tag{1.3}$$

We now vary both sides of (1.3) in the neighbourhood of the state $\lambda = \lambda_{(0)}$, $x_0 = x_{(0)0}$, $x_0^{\cdot} = x_{(0)0}^{\cdot}$

$$\begin{gathered} \delta x_{(0)0} = \frac{\partial x(T)}{\partial x_0}\delta x_{(0)0} + \frac{\partial x(T)}{\partial x_0^{\cdot}}\delta x_{(0)0}^{\cdot} + \frac{\partial x(T)}{\partial \lambda}\delta\lambda_{(0)} \\ \delta x_{(0)0}^{\cdot} = \frac{\partial x^{\cdot}(T)}{\partial x_0}\delta x_{(0)0} + \frac{\partial x^{\cdot}(T)}{\partial x_0^{\cdot}}\delta x_{(0)0}^{\cdot} + \frac{\partial x^{\cdot}(T)}{\partial \lambda}\delta\lambda_{(0)} \\ (x_{(0)0} = x_{(0)}(0)) \end{gathered} \tag{1.4}$$

Introducing the notation $\partial x(t)/\partial x_0 = y_1(t)$, $\partial x(t)/\partial x_0^{\cdot} = y_2(t)$, $\partial x(t)/\partial\lambda = y_\lambda(t)$, we determine $y_1(T)$, $y_2(T)$, $y_\lambda(T)$ from the solutions of the appropriate variational equations

$$\begin{gathered} y_1^{\cdot\cdot} + k^2 y_1 \cos x = 0, \quad y_{10} = 1, \quad y_{10}^{\cdot} = 0 \\ y_2^{\cdot\cdot} + k^2 y_2 \cos x = 0, \quad y_{20} = 0, \quad y_{20}^{\cdot} = 1 \end{gathered} \tag{1.5}$$

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